October Math Gems

Problem of the week 20

§1 problems

Problem 1.1. Find all real numbers x, y, z so that

$$x^2y + y^2z + z^2 = 0$$
$$z^3 + z^2y + zy^3 + x^2y = \frac{1}{4}(x^4 + y^4).$$

Problem 1.2. If x is a real number satisfying the equation

$$9\log_3 x - 10\log_9 x = 18\log_{27} 45,$$

then the value of x is equal to $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

Problem 1.3. Find all primes p, such that there exist positive integers x, y which satisfy

$$\begin{cases} p + 49 = 2x^2 \\ p^2 + 49 = 2y^2 \end{cases}$$

Problem 1.4. Suppose x and y are real numbers satisfying

$$\begin{cases} x^3 - y^3 = 493. \\ x^2y - y^2x = 50. \end{cases}$$

What is the positive difference between x and y?

Problem 1.5. Find all pairs (x, y) of real numbers satisfying the system : $\begin{cases} x + y = 2 \\ x^4 - y^4 = 5x - 3y \end{cases}$

Problem 1.6. Solve the equation in $\mathbb{R}$, the system $\begin{cases} x + y + xy = 4 \\ y + z + yz = 7 \\ x + z + xz = 9 \end{cases}$

Problem 1.7. If a dan b are positive numbers and satisfy, ${}^a log 4 ={}^b log 10 ={}^{a-b} log 25$ What are the value of a and b?

Problem 1.8. Solve

$$\log_x\left(\frac{x^{4x-6}}{2}\right) = 2x - 3.$$

Problem 1.9. Let a, b, and c be distinct positive integers such that $\sqrt{a}+\sqrt{b}=\sqrt{c}$ and c is not a perfect square. What is the least possible value of $a+b+c$?

Problem 1.10. Solve this system of equations

$$\begin{cases} x^2 = y^3+1 \\ y^2 = x^3 - 23 \end{cases}$$

Problem 1.11. Starting with a 5×5 grid, choose a 4×4 square in it. Then, choose a 3×3 square in the 4×4 square, and a 2×2 square in the 3×3 square, and a 1×1 square in the 2×2 square. Assuming all squares chosen are made of unit squares inside the grid. In how many ways can the squares be chosen so that the final 1×1 square is the center of the original 5×5 grid?

Problem 1.12. Let $ABCD$ be a rectangle with $AB=10$ and $AD=5$. Suppose points P and Q are on segments CD and BC, respectively, such that the following conditions hold: $BD \parallel PQ$ $\angle APQ = 90^\circ$. What is the area of $\triangle CPQ$?

Problem 1.13. How many real roots does this log equation have?

$$\log_{(x^2-3x)^3} 4 = \frac{2}{3}$$

Should I use the fundamental theorem of algebra for this problem?

Problem 1.14. In trapezoid $ABCD$, leg $\overline{BC}$ is perpendicular to bases $\overline{AB}$ and $\overline{CD}$, and diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. Given that $AB=\sqrt{11}$ and $AD=\sqrt{1001}$, find BC^2.

Problem 1.15. Find

$$\cos\frac{2\pi}{2013}+\cos\frac{4\pi}{2013}+\cdots+\cos\frac{2010\pi}{2013}+\cos\frac{2012\pi}{2013}$$

Problem 1.16. Find all triples (a,b,c) of real numbers such that $ab+bc+ca=1$ and

$$a^2b+c=b^2c+a=c^2a+b.$$

Problem 1.17. Solve over $\mathbb{R}$ the equation $4^{(sinx)^2}+3^{(tanx)^2}=4^{(cosx)^2}+3^{(cotanx)^2}$.

Problem 1.18. Two distinct, real, infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is $1/8$, and the second term of both series can be written in the form $\frac{\sqrt{m}-n}{p}$, where m, n, and p are positive integers and m is not divisible by the square of any prime. Find $100m+10n+p$.

Problem 1.19. Let a,b,c, and d be real numbers that satisfy the system of equations

$$\begin{aligned} a+b &= -3 \\ ab+bc+ca &= -4 \\ abc+bcd+cda+dab &= 14 \\ abcd &= 30. \end{aligned}$$

There exist relatively prime positive integers m and n such that

$$a^2+b^2+c^2+d^2=\frac{m}{n}.$$

Find $m+n$.

Problem 1.20. An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$ where m and n are relatively prime integers. Find $m+n$.